

# Solitary and Jacobi elliptic wave solutions of the generalized Benjamin-Bona-Mahony equation

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## Abstract

Exact bright, dark, antikink solitary waves and Jacobi elliptic function solutions of the generalized Benjamin-Bona-Mahony equation with arbitrary power-law nonlinearity will be constructed in this work. The method used to carry out the integration is the F-expansion method. Solutions obtained have fractional and integer negative or positive power-law nonlinearities. These solutions have many free parameters such that they may be used to simulate many experimental situations, and to precisely control the dynamics of the system.

Keywords: Generalized Benjamin-Bona-Mahony equation, F-expansion method, solitary waves, Jacobi elliptic function solutions.

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## 1. Introduction

The search of exact solutions for evolution nonlinear partial differential equations (NLPDEs) plays an important role in the study of nonlinear physical phenomena. This is due to the fact that, nonlinear phenomena are ubiquitous in nature, thus appear in a wide range of fields in physics, mathematical physics, and engineering. Some celebrated evolution NLPDEs are the Hasegawa-Mima equation [1] which describes turbulence in plasma physics, the Fitzhugh-Nagumo equation [2] that models biological neuron, the Hunter-Saxton equation [3-4] used to study waves orientation in nematic liquid crystal, the nonlinear Schrödinger equation that models the dynamics of waves in many media such as matter waves in Bose-Einstein condensates [5], the evolution of electromagnetic fields in fiber optics [6], the evolution of gravity driven surface water waves [7], the evolution of the order parameter in the BCS theory [8], just to name a few.

In the past five decades, a great deal of attention has been paid to the dynamics of shallow water waves, mainly modeled by an evolution NLPDE known as the Korteweg-de Vries (KdV) equation [9], and modified KdV equations [9]. The KdV equation is valid when the water depth is constant and is derived under the assumption of small wave-amplitude and large wave length. Modified KdV equations include KdV equations with varying bottom and higher order corrections of the KdV equation. Solutions of the KdV and modified KdV equations have actively been investigated, and include solitary waves which come from a delicate balance between dispersion and nonlinearity, periodic waves like the Jacobi elliptic function solutions and so on [9,10].

In 1972, the regularized long-wave equation, better known as the Benjamin-Bona-Mahony (BBM) equation [11] were introduced as a regularized form of the KdV equation. As pointed out in [11,12], the BBM equation better describes long waves and, as far as the existence, uniqueness and stability are concerned, the BBM equation has some substantial advantages over the KdV equation [11]. Moreover, the BBM equation also finds applications in other contexts such as the modeling of the drift of waves in plasma physics or the Rossby waves in rotating fluids [13], wave transmission in semi-conductors and optical devices [14], hydromagnetic waves in cold plasma, acoustic waves in anharmonic crystals, and acoustic gravity waves in compressible fluids [11]. In the past four decades, the BBM equation and its various versions have been intensively studied, and many types of solutions have been

found among them solitary waves and periodic waves which can be found in the explicit literature. A classification of some forms of the BBM equation and their solutions can be found in [15]. Unlike the KdV equation, the BBM equations are not exactly integrable in the sense of the Painlevé test of integrability. Nevertheless, various techniques have been developed which help to carry out the integration of BBM equations. Among them are the tanh and the sine-cosine methods [16], the Jacobi elliptic function expansion method [17], the first integral method [18], the variable-coefficient balancing-act method [19], the hyperbolic auxiliary function method [20], the homogeneous balance method [21]. A generalized (1+1) BBM equation with dual arbitrary power-law nonlinearity may be written in dimensionless form as [22,23]

$$u_t + \alpha u_x + (\beta u^n + \gamma u^{2n})u_x - \delta u_{xxt} = 0, \quad (1)$$

where  $u$  is the wave profile,  $\alpha$  and  $\delta$  are the dispersion coefficients,  $n$  is the arbitrary power-law nonlinearity,  $\beta$  and  $\gamma$  the coefficients of the dual power-law nonlinearity. For  $\alpha = 1$ ,  $\beta = 1$ ,  $n = 1$ ,  $\gamma = 0$ , and  $\delta = 1$  Eq.(1) recovers the BBM equation. Wazwaz solved Eq.(1) in the case where  $\alpha = 1$ ,  $\delta = -1$ ,  $\beta = 0$  by means of the tanh and the sine-cosine methods, and obtained solitary and periodic solutions [24]. Yang, Tang, and Qiao constructed solitary and periodic wave solutions of Eq.(1) for  $\alpha = 0$ ,  $n > 0$  and  $\delta \neq 0$  using an improved tanh function method [22]. Liu, Tian, and Wu used the Weierstrass elliptic function method to construct two solutions of Eq.(1) for  $\alpha = 0$  in terms of the Weierstrass elliptic functions [25]. Biswas employed the solitary wave ansatz method and proposed a one-soliton solution of Eq.(1) [23]. All the latter works show the importance of investigating solutions of the [BBM equation given by Eq.\(1\)](#). However, to the best of our knowledge, Eq.(1) with non-vanishing coefficients has only been tackled in the work of Ref. [23] where a one-soliton solution was found. As an evolution NLPDE with important applications in different fields in physics, it is important to find more solutions of Eq.(1) that may help to have a better understanding of physical phenomena or at least give orientations for future applications. For example, solitary and Jacobian elliptic function solutions have been intensively used for practical applications in physics and engineering; these solutions have not been fully investigated for the generalized BBM equation ([Eq.\(1\)](#)) with all non-vanishing coefficients.

The aim of this work is to construct solitary and Jacobi elliptic wave solutions of the generalized BBM equation ([Eq.\(1\)](#)) with all non-vanishing coefficients. To this end, we

use the F-expansion method introduced in [26] which has been an accurate tool to integrate evolution NLPDEs, along with the auxiliary ordinary equation [5,27]. The paper is organized as follows, in Sec. 2, we construct analytical solutions of Eq.(1). Then we discuss the characteristics and evolution of the solutions in Sec. 3. The paper is concluded in Sec. 4.

## 2. Analytical solutions

We start our quest of analytical solutions of Eq.(1) by setting the following traveling wave transformation

$$u(x, t) = U(\zeta), \quad \zeta = kx - Vt, \quad (2)$$

where  $k$  is the inverse of the width of the wave and  $V$  its velocity. Inserting Eq.(2) into Eq.(1) we obtain a nonlinear ordinary differential equation for the function  $U$

$$(\alpha k - V)U_\zeta + k(\beta U^n + \gamma U^{2n})U_\zeta + \delta k^2 V U_{\zeta\zeta\zeta} = 0, \quad (3)$$

with  $U_\zeta \equiv \frac{\partial U}{\partial \zeta}$  and  $U_{\zeta\zeta\zeta} \equiv \frac{\partial^3 U}{\partial \zeta^3}$ . An integration of Eq.(3) yields

$$(\alpha k - V)U + k\left(\frac{\beta}{n+1}U^{n+1} + \frac{\gamma}{2n+1}U^{2n+1}\right) + \delta k^2 V U_{\zeta\zeta} = 0, \quad (4)$$

where the right-hand side constant of integration has been set to zero. Eq.(4) is difficult to solve analytically, in order to find analytical solutions, we need to transform it into a more tractable and manageable form. Toward that end, we use the transformation  $\omega = U^n$ . After a little algebra, a nonlinear ordinary differential equation in terms of the function  $\omega$  is retrieved

$$\omega\omega_{\zeta\zeta} + p\omega^2 + q\omega^3 + r\omega^4 + s(\omega_\zeta)^2 = 0, \quad (5)$$

in which the parameters  $p, q, r, s$  are given by

$$p = \frac{(\alpha k - V)n}{\delta k^2 V}, \quad (6a)$$

$$q = \frac{\beta n}{\delta k(n+1)V}, \quad (6b)$$

$$r = \frac{\gamma n}{\delta k(2n+1)V}, \quad (6c)$$

$$s = \frac{1-n}{n}, \quad (6d)$$

with these forbidden values for  $n$ :  $-1, -1/2, 0, 1$ .

The next step is to solve Eq.(5) by means of the F-expansion method. Thus, following the standard procedure [5,26,27], we seek the solution in the form

$$\omega(\zeta) = \sum_{i=0}^N a_i F^i(\zeta), \quad (7)$$

where the function  $F$  satisfies the auxiliary equation

$$\left(\frac{dF}{d\zeta}\right)^2 = b_0 + b_1 F(\zeta) + b_2 F^2(\zeta) + b_3 F^3(\zeta) + b_4 F^4(\zeta). \quad (8)$$

Exact solutions of the auxiliary equation (8) can be found in Ref. [27], while the coefficients  $a_i, b_j$  ( $j = 0, 1, 2, 3, 4$ ) will be determined later. The value of the integer  $N$  is found by balancing the highest nonlinear and derivative terms. This idea in our case leads to  $N = 1$ . Hence,

$$\omega(\zeta) = a_0 + a_1 F(\zeta). \quad (9)$$

Inserting Eq.(9) along with Eq.(8) into Eq.(5), and collecting all the coefficients of powers of  $F$  ( $F^m(\zeta)$ ,  $m = 0, 1, 2, 3, 4$ ), setting each coefficient to zero, yields a set of algebraic equations for the unknowns  $a_0, a_1, q, r, b_j$  ( $j = 0, 1, 2, 3, 4$ ) and or  $s$

$$a_0 a_1 b_1 + 2p a_0^2 + 2q a_0^3 + 2r a_0^4 + 2s a_1^2 b_0 = 0, \quad (10a)$$

$$2a_0 a_1 b_2 + a_1^2 b_1 + 4p a_0 a_1 + 6q a_0^2 a_1 + 8r a_0^3 a_1 + 2s a_1^2 b_1 = 0, \quad (10b)$$

$$2a_1^2 b_2 + 3a_0 a_1 b_3 + 2p a_1^2 + 6q a_0 a_1^2 + 12r a_0^2 a_1^2 + 2s a_1^2 b_2 = 0, \quad (10c)$$

$$4a_0 a_1 b_4 + 3a_1^2 b_3 + 2q a_1^3 + 8r a_0 a_1^3 + 2s a_1^2 b_3 = 0, \quad (10d)$$

$$2a_1^2 b_4 + r a_1^4 + s a_1^2 b_4 = 0. \quad (10e)$$

As explained in [27], the solutions of the auxiliary equation are sensitive to specific values of the coefficients  $b_j$  ( $j = 0, 1, 2, 3, 4$ ) and can be regrouped into two families: family (I)  $b_0 = b_1 = 0$ ; family (II)  $b_1 = b_3 = 0$ . We show below that these two families lead to solitary waves and Jacobi elliptic function solutions of Eq.(1).

## 2.1 Family I solutions: $b_0 = b_1 = 0$

We simplify the system of Eqs.(10) by setting for example  $p = -1$  and obtain the following solutions:

$$a_{01} = 0, a_{11} = a_{11}, b_{21} = \frac{1}{s+1}, q_1 = -\frac{b_3(2s+3)}{2a_{11}}, b_{41} = -\frac{a_{11}^2 r}{s+2}, \quad (11a)$$

$$a_{02} = \sqrt{\frac{-(s+2)}{r(s+1)}}, a_{12} = \frac{b_3}{2} a_{02}, b_{22} = b_{21}, q_2 = \frac{(2s+3)}{2a_{02}}, b_{42} = \frac{-b_3^2(s+1)}{4}, \quad (11b)$$

$$a_{03} = -a_{02}, a_{13} = -a_{12}, b_{23} = b_{21}, q_3 = -q_2, b_{43} = b_{42}. \quad (11c)$$

Therefore, it is possible to obtain solitary wave solutions if we consider these solutions of the auxiliary equation (8):

$$b_0 = b_1 = 0, b_2 > 0, F_{I,1}(\zeta) = \frac{-b_2 b_3 \text{sech}^2(\sqrt{b_2}/2)\zeta}{b_3^2 - b_2 b_4 [1 - \tanh(\sqrt{b_2}/4\zeta)]^2}, \quad (12a)$$

$$b_0 = b_1 = 0, b_2 > 0, b_3^2 - b_2 b_4 > 0, F_{I,2} = \frac{2b_2 \text{sech}(\sqrt{b_2}\zeta)}{\sqrt{b_3^2 - 4b_2 b_4} - b_3 \text{sech}(\sqrt{b_2}\zeta)}. \quad (12b)$$

Hence, the analytical solution of Eq.(1) is written as

$$u_{Im}(x, t) = (a_{0m'} + a_{1m'} F_{Im}(\zeta))^{\frac{1}{n}}, \quad (13)$$

where  $m = 1, 2$  and  $m' = 1, 2, 3$ . Solutions given by Eq.(13) are correct ones if that of the set (11) are compatible with the requirements of the parameters  $b_j$  of the set of Eq.(12). These compatibility conditions impose some restrictions to our solutions Eq.(13). To be precise, in the set of Eqs.(12),  $b_2 > 0$  implies  $n > 0$ . This means that our solutions correspond to only positive values of the power-law nonlinearities (in addition to  $n \neq 1$  obtained above). When  $m = 2$  in Eq.(13), the condition  $b_3^2 - 4b_2 b_4 > 0$  imposes  $|b_3| > 2n |a_1| \sqrt{\frac{-r}{n+1}}$ ; without loss of generality, we consider that  $k, V > 0$  and taking into account the expression of  $r$  given by Eq.(6c),  $r$  must be negative, hence  $\gamma$  and  $\delta$  have opposite signs. Moreover, other important information can be extracted from the set of Eqs.(11). On the first hand, after an examination of Eq.(11a) (zero background solutions  $a_{01} = 0$ ), one deduces that  $a_1, b_3, \gamma, \delta, k$ , and  $V$  are free parameters, while  $\alpha = \frac{V(1-\frac{\delta k^2}{n})}{k}$  and  $\beta = \frac{-b_3 k V (n+2)(n+1)\delta}{2n^2 a_1}$ . The experimenter has many possibilities to manipulate some physical parameters of the solution like its width, velocity, amplitude, and to consider different physical situations by playing with the values and signs of  $\gamma, \delta$ , and  $n$ , meanwhile altering the signs and values

of  $\alpha$ ,  $\beta$ . For  $m = 1$  in Eq.(13), solitary wave solutions are obtained when  $\delta$  and  $\gamma$  have the same signs, a situation consistent with the existence of solitary waves in many physical system due to a balance between dispersion and nonlinearity [5,6] (and references therein). Additional specific conditions lead to bright or dark solitary waves. A bright solitary wave is obtained in the following cases: (i)  $a_1$  and  $b_3$  with opposite signs and  $n$  is an integer, a sample is presented in Fig. 1(a) for  $n = 2, m' = 1$ ; (ii)  $n = 1/n'$ ,  $n'$  even, an example is depicted in Fig. 1(b) where  $n' = 2, m' = 1$ ; (iii)  $a_1, b_3$  with opposite signs and  $n = 1/n'$ ,  $n'$  odd, Fig. 1(c) ( $n' = 3, m' = 1$ ) displays such a case. A black solitary wave solution is obtained if  $a_1$  and  $b_3$  have opposite signs and  $n = 1/n'$  with  $n'$  odd as can be seen in Fig. 1(d) ( $n' = 3, m' = 1$ ). On the second hand, we observe that Eqs.(11b) and (c) may give rise to solitary wave solutions on finite backgrounds since  $a_0$  is real and does not vanish,  $\beta = \pm \frac{kV\delta(n+2)(n+1)}{2n^2} \sqrt{\frac{-r}{n}}$ . For  $m = 1$  in Eq.(13), compatible conditions on  $r$  and  $b_2$  are identical to the case of Eq.(11a), while for  $m = 2$ , the compatible condition  $b_3^2 - 4b_2b_4 > 0$  is always satisfied ( $b_3^2 - 4b_2b_4 = 2b_3^2$ ). We now present solitary waves embedded on finite backgrounds. To this end, we focus on the solution Eq.(13) with  $m = 2$  and recall that  $\delta$  and  $\gamma$  must have opposite signs. Bright solitary waves are obtained if  $b_3 > 0$ ,  $n = 1/n'$  and (i)  $n'$  even (see Fig. 2(a)  $n' = 2, m' = 3$ ), (ii)  $n'$  odd with  $a_0, a_1 > 0$  (see Fig. 2(b)  $n' = 3, m' = 3$ ). A dark solitary wave is obtained with the conditions (ii) but  $a_0, a_1 < 0$  (see Fig. 2(c)  $n' = 3, m' = 2$ ). For  $b_3 < 0$ , Eq.(13) represents a dark solitary wave in many cases, but in some cases where the background is sufficiently small and negative,  $n'$  being an odd integer, an 'antibright' solitary wave solution can be obtained (Fig. 2(d)  $n' = 3, m' = 2$ ).

## 2.2 Family II solutions $b_0 = b_1 = 0$

In this part of the work, we show that Eq.(1) also admits Jacobi elliptic function solutions. The solutions of the set of Eqs.(10) for  $b_1 = b_3 = 0$  are

$$b_{01} = \frac{a_{04}^2(p + 4a_{04}^2r)}{4a_{14}^2}, b_{24} = 2a_{04}^2r - \frac{p}{2}, q_4 = \frac{-4a_{04}^2r + p}{2a_{04}}, \quad (14a)$$

$$b_{44} = \frac{a_{14}^2(p - 4a_{04}^2r)}{a_{04}^2}, s = \frac{-2(p - 2a_{04}^2r)}{p - 4a_{04}^2r}. \quad (14b)$$

Taking advantage of four Jacobi elliptic function solutions of the auxiliary Eq.(8):

$$b_{01} = \frac{1}{4}, b_{24} = \frac{k_1^2 - 2}{2}, \quad (15a)$$

$$F_{II1}(\zeta) = \frac{\text{cn}(\zeta, k_1)}{\sqrt{1 - k_1^2 + \text{dn}(\zeta, k_1)}}, F_{II2}(\zeta) = \frac{\text{cn}(\zeta, k_1)}{\sqrt{1 - k_1^2 - \text{dn}(\zeta, k_1)}}, \quad (15b)$$

$$F_{II3}(\zeta) = \frac{\text{sn}(\zeta, k_1)}{1 + \text{dn}(\zeta, k_1)}, F_{II4}(\zeta) = \frac{\text{sn}(\zeta, k_1)}{1 - \text{dn}(\zeta, k_1)}, \quad (15c)$$

the Jacobi elliptic function solutions of Eq.(1) are

$$u_{II m}(x, t) = (a_{0m'} + a_{1m'} F_{II m}(\zeta))^{\frac{1}{n}}, \quad (16)$$

where  $m = 1, 2, 3, 4$ ,  $k_1$  ( $0 < k_1 < 1$ ) denotes the modulus of Jacobi elliptic functions. As already explained above, correct solutions of Eq.(16) require that the parameters  $b_j$  in Eq.(15a) be compatible with their counterparts of the set of Eqs.(14), providing the expressions of  $a_0, a_1, p, k_1$

$$a_{04} = \sqrt{\frac{-(n+1)}{3nr}}, a_{14} = \frac{1}{3} \sqrt{\frac{-(n+1)}{nr}}, p = \frac{-4}{3n}, k_1 = \frac{\sqrt{6}}{3}, \quad (17a)$$

$$a_{05} = a_{04}, a_{15} = -a_{14}, p, k_1, \quad (17b)$$

$$a_{06} = -a_{04}, a_{16} = a_{14}, p, k_1, \quad (17c)$$

$$a_{07} = -a_{04}, a_{17} = -a_{14}, p, k_1. \quad (17d)$$

It is simple to deduce from the set of Eqs.(17) that in Eq.(16)  $m' = 4, 5, 6, 7$ ,  $\alpha = \frac{V}{k}(\frac{-4\delta k^2}{3n} + 1)$ , and  $\beta = \frac{\delta k(n+1)V q_4}{n}$ , while  $q_4$  is given in Eq.(14). The parameters  $n, \gamma, \delta, k, V$  are free in solutions Eq.(16). In order to represent some periodic solutions, we suppose that  $k, V, n > 0$ , consequently,  $\gamma$  and  $\delta$  have opposite signs since  $r$  should be negative in the set of Eqs.(17). Furthermore, the modulus  $k_1 = \sqrt{6}/3$  is always constant. We plot in Fig 3 two periodic solutions of Eq.(1); in panel (a)  $n = 2, m = 1, m' = 7$ , while in panel (b)  $n = 2, m = 3, m' = 7$ .

### 3. Discussion

In [23] first appeared a bright solitary wave solution of Eq.(1) with all non null coefficients, the arbitrary power-law nonlinearity  $n$  belonging to the interval  $0 < n < 2$ . In Sec. 2, we have proposed bright, black, dark, and Jacobi elliptic function solutions of Eq.(1) with



all non-vanishing coefficients, and show that the power-law nonlinearity  $n$  is allowed to take more positive values than in [23] ( $n > 0, n \neq 1$ ). Hence, the solutions presented in Sec. 2 may be used to probe the validity of the model (Eq.(1)) with higher power-law nonlinearities. Moreover, the two families of solutions also admit negative values of the power-law nonlinearity  $n$ . For example, Jacobi elliptic function solutions given by Eq.(16) allow negative  $n$  in the following cases: (i)  $-1 < n < 0, r > 0$  which means that  $\delta, \gamma$  have the same signs; (ii)  $n < -1, r < 0$  thus  $\delta, \gamma$  have opposite signs. We display in Fig. 4(a) the solution expressed by Eq.(16) for  $n = -3, m = 3, m' = 4$ . Besides, the solution Eq.(13) for negative  $n$  holds only for  $m = 1$  (the case  $m = 2$  does not satisfy the condition  $b_3^2 - 4b_2b_4 > 0$ ). Setting  $p = 1$ , the solutions of the set of Eqs.(10) with  $n < 0$  become  $a_{01} = 0, a_{11} = a_{11}, b_{21} = \frac{-1}{s+1}, q_1 = -\frac{b_3(2s+3)}{2a_{11}}, b_{41} = -\frac{a_{11}^2 r}{s+2}, a_{02} = \sqrt{\frac{(s+2)}{r(s+1)}}, a_{12} = \frac{-b_3}{2}a_{02}, b_{22} = b_{21}, q_2 = \frac{-(2s+3)}{2a_{02}}, b_{42} = \frac{-b_3^2(s+1)}{4}, a_{03} = -a_{02}, a_{13} = -a_{12}, b_{23} = b_{21}, q_3 = -q_2, b_{43} = b_{42}$ . One may expect that the solution Eq.(13) gives rise to rather different solitary wave solutions since the physics of the model has completely changed. Such an expectation is confirmed by Fig. 4(b) where an antikink profile solution is presented for  $n = -1/3, m = 1, m' = 2$ .

#### 4. Conclusion

This paper obtains bright, dark, antikink, 'antibright' solitary waves and Jacobi elliptic function solutions of the generalized BBM equation with dual power-law nonlinearity. The F-expansion method (along with the auxiliary equation) is used to integrate the BBM equation. Solutions are constructed for arbitrary power-law nonlinearities that might be positive or negative, fractional or integer. Many physical parameters may be used to mimic a wide range of experimental scenarios, whereas the solutions constructed may be used to probe the accuracy of the model with higher power-law nonlinearities. The issue of stability of the solutions of the generalized BBM equation needs a thorough treatment that shall be addressed by means of perturbation techniques and numerical simulations in future works.

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## FIGURE CAPTIONS

### Figure 1

(Color online) Spatiotemporal evolution of solitary wave solutions with positive power-law nonlinearity given by Eq.(13). (a)  $n = 2$ ,  $\alpha = -1$ ,  $\beta = 0.1$ ,  $\gamma = 2$ ,  $\delta = 2$ ,  $V = 1$ ,  $k = 1$ ,  $b_3 = 0.1$ ,  $a_1 = -1$ . (b) Same parameters as in panel (a) except  $n = 1/2$  and  $a_1 = 1$ . (c) Same parameters as in panel (a) except  $n = 1/3$ ,  $\beta = 0.28$ . (d) Same parameters as in panel (b) except  $n = 1/3$ ,  $\beta = -0.28$ . (a)-(c) bright solitary waves, (d) black solitary wave.

### Figure 2

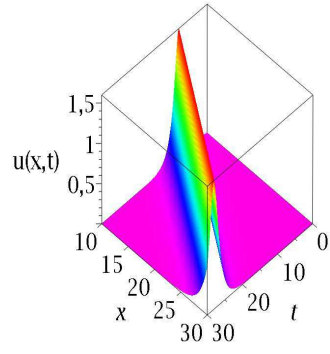
(Color online) (a)-(b) bright, (c) dark solitary wave solutions (13) on finite backgrounds. (a)  $n = 1/2$ ,  $\alpha = 2$ ,  $\delta = -1$ ,  $\beta = 5\sqrt{3}$ ,  $a_0 = \sqrt{3}$ ,  $a_1 = 0.05\sqrt{3}$ ,  $m' = 3$ . (b)  $n = 1/3$ ,  $\beta = 2.8\sqrt{30}$ ,  $a_0 = \sqrt{30}/3$ ,  $a_1 = 0.01666\sqrt{30}$ . (c)  $n = 1/3$ ,  $\beta = -0.28\sqrt{30}$ ,  $a_0 = -\sqrt{30}/3$ ,  $a_1 = -0.01666\sqrt{30}$ ,  $m' = 2$ . Other parameters in (a)-(c) are the same as in Fig. 1(a). (d)  $a_1 = 0.01666\sqrt{30}$ , other parameters as in (c).

### Figure 3

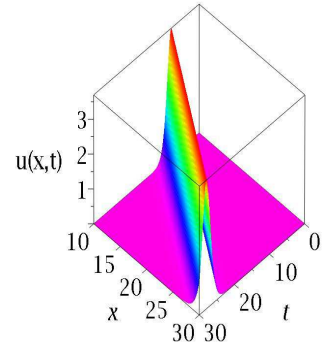
(Color online) Spatiotemporal evolution of Jacobi elliptic function solutions (16). (a)  $m = 1$ , (b)  $m = 3$ . Parameters are  $n = 2$ ,  $\alpha = 2/3$ ,  $\beta = 4\sqrt{5}/5$ ,  $\gamma = -1$ ,  $\delta = 1$ ,  $a_0 = \sqrt{5}/2$ ,  $a_1 = \sqrt{15}/6$ ,  $m' = 7$ . Other parameters as in Fig 1(a).

### Figure 4

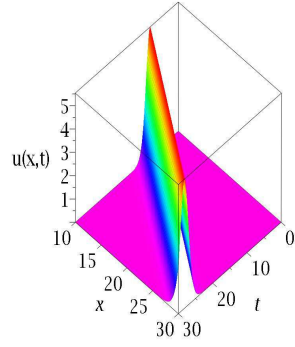
(Color online) Solutions with negative power-law nonlinearity. (a) Jacobi elliptic function solution (16),  $n = -3$ ,  $\alpha = 23/27$ ,  $\beta = \frac{2}{45}\sqrt{30}$ ,  $a_0 = \sqrt{30}/9$ ,  $a_1 = \sqrt{10}/9$ ,  $m' = 4$ , other parameters as in Fig. 3(b). (b) Antikink solution (13),  $n = -1/3$ ,  $\alpha = 4$ ,  $\beta = 10\sqrt{3}$ ,  $\delta = -1$ ,  $\gamma = 2$ ,  $a_0 = -\sqrt{3}/3$ ,  $a_1 = 0.0166\sqrt{3}$ ,  $m = 2$ ,  $m' = 1$ .



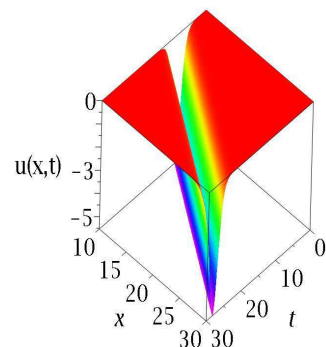
(a)



(b)



(c)



(d)

FIG. 1:

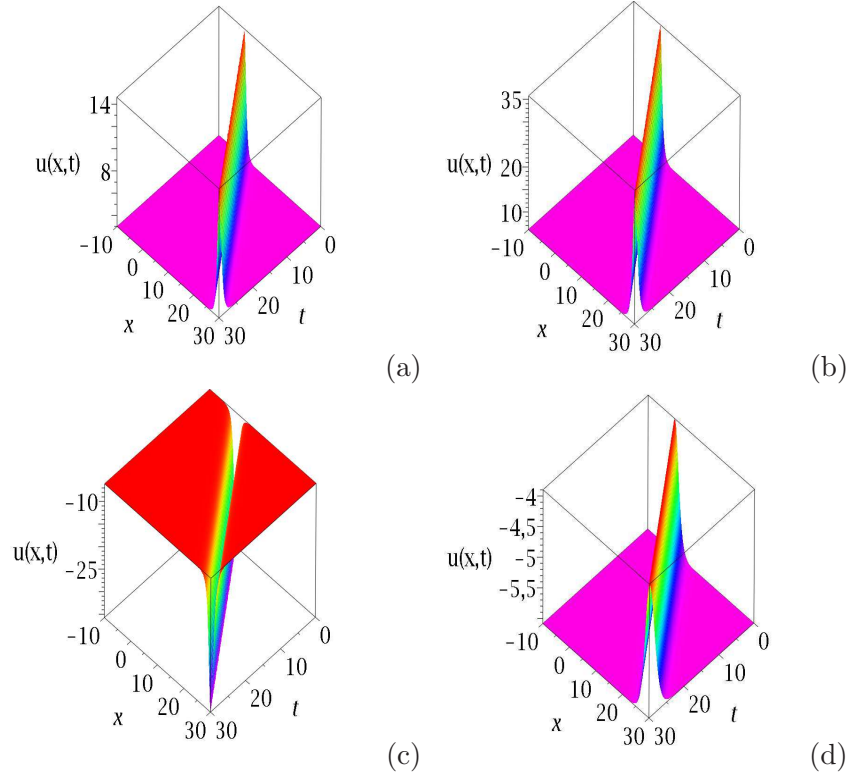


FIG. 2:

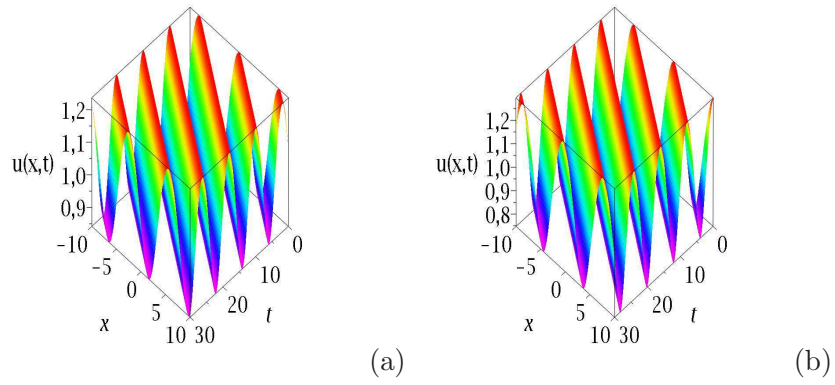
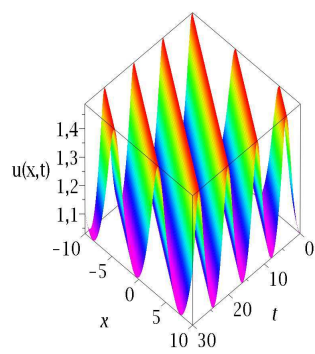
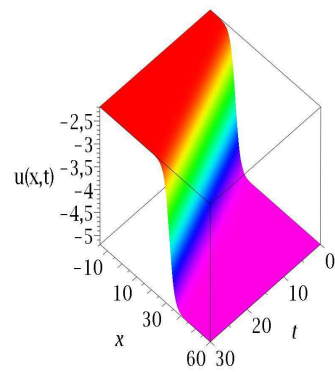


FIG. 3:



(a)



(b)

FIG. 4: